Project Dynamical Systems

23 June 2017

university of groningen

Exam Problem Sheet

The exam consists of 5 problems. You have 120 minutes to answer the questions. Give brief but precise answers. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**3+3+3 Points.**]

   Each of the following one-dimensional systems depends on a parameter $a \in \mathbb{R}$. Describe the bifurcations involved, sketch the corresponding bifurcation diagrams, and classify the bifurcations.

   (a) $x' = x^2 - ax$

   (b) $x' = x^3 - ax$

   (c) $x' = x^3 - x + a$

2. [**9 Points.**]

   Consider the planar system

   $$X' = \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix} X .$$

   Sketch the regions in the $a-b$ plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. [**3+3+3+2 Points.**]

   Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Then the system

   $$X' = -\nabla V(X) \qquad (1)$$

   is called a *gradient system* (here $\nabla V(X) = (\frac{\partial}{\partial x_1}V(X), \dots, \frac{\partial}{\partial x_n}V(X))$ with $X = (x_1, \dots, x_n) \in \mathbb{R}^n$). It is clear that the equilibrium points of a gradient system are given by the critical points of V.

   (a) Show that if X is not an equilibrium point, then V is strictly decreasing along the solution curve through X.

– **please turn over** –

(b) State the definition of asymptotic stability for the equilibrium point of a time continuous system.

(c) Show that if X^* is an isolated minimum of V then X^* is asymptotically stable. What can you say about the basin of attraction of X^* in this case?

(d) What are the conditions on V at an equilibrium point X^* to conclude stability from the linearization of the gradient system at X^*?

4. [**4+4 Points.**]
Consider the planar system

$$r' = r - r^2,$$
$$\theta' = 1,$$

where (r, θ) are polar coordinates.

(a) For each point in the plane, identify the α and ω limit sets.

(b) State the Poincaré-Bendixson theorem and use it to show that the system has a limit cycle.

5. [**4+4 Points.**]
Consider the discrete time system $x_{n+1} = f_\lambda(x_n)$, where $f_\lambda : \mathbb{R} \to \mathbb{R}$ for each parameter $\lambda \in \mathbb{R}$ and $f_\lambda(x)$ is a smooth function of x and λ. Prove that if the system has a fixed point x^* for λ_0 with $|f'_{\lambda_0}(x^*)| < 1$, then there is an interval I about x^* and an interval J about λ_0 such that, if $\lambda \in J$, then

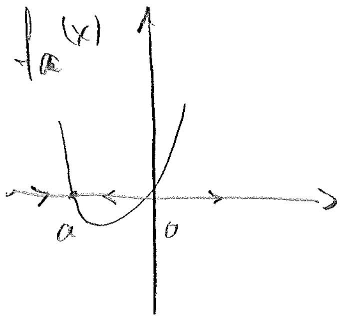
(a) f_λ has a unique fixed point which is a sink in I, and

(b) all orbits $x_{n+1} = f_\lambda(x_n)$ with starting point $x_0 \in I$ converge to x^* for $n \to \infty$.

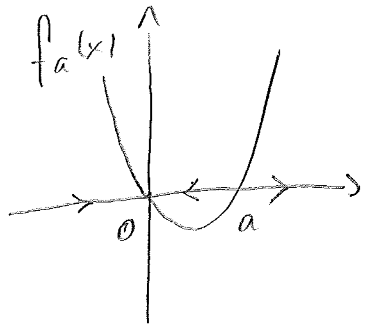
1.

(a) $x' = x^2 - ax$
$= x(x-a) =: f_a(x)$

$a<0$:

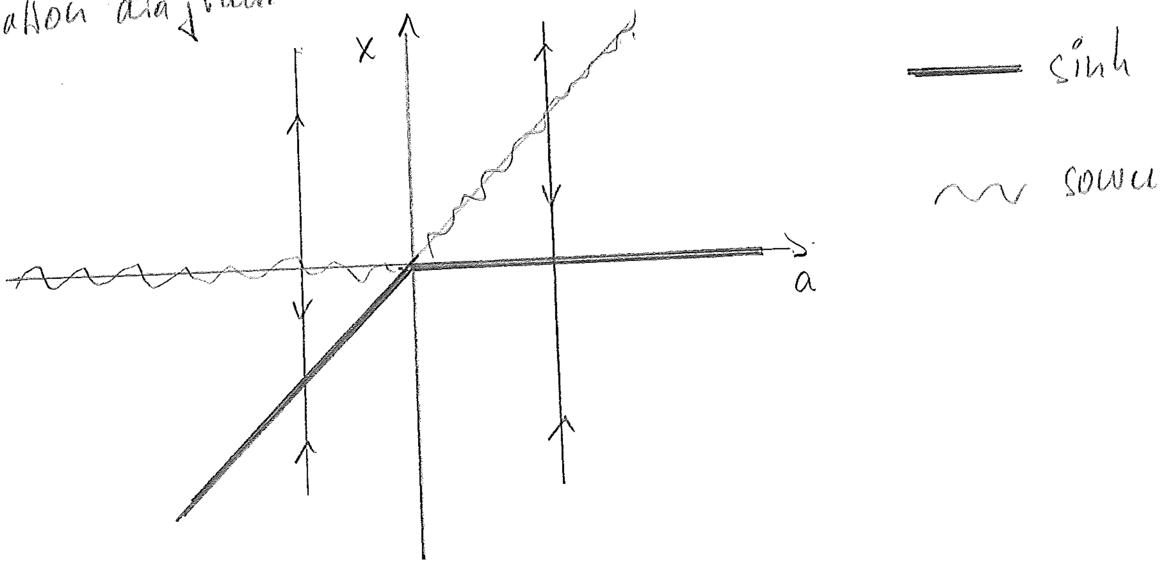


$a>0$:



equilibria: $x=0$ sink for $a>0$, source for $a<0$
$x=a$ sink for $a<0$, source for $a>0$

bifurcation diagram

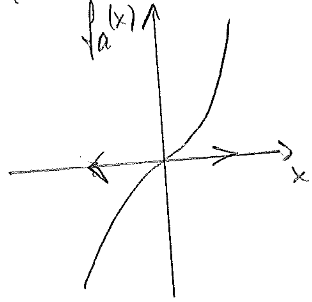


this is a transcritical bifurcation.
(two equilibria traverse and exchange their stability)

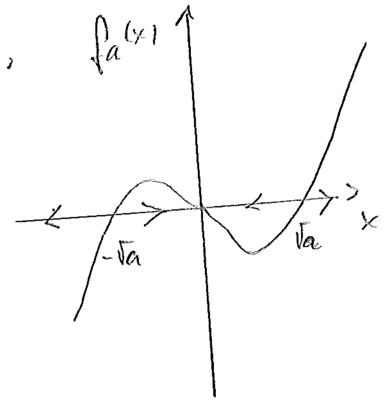
(b) $x' = x^3 - ax$
$= x(x^2 - a) =: f_a(x)$

equilibria: $x = 0$ and $x = \pm\sqrt{a}$ for $a > 0$

$a < 0$:



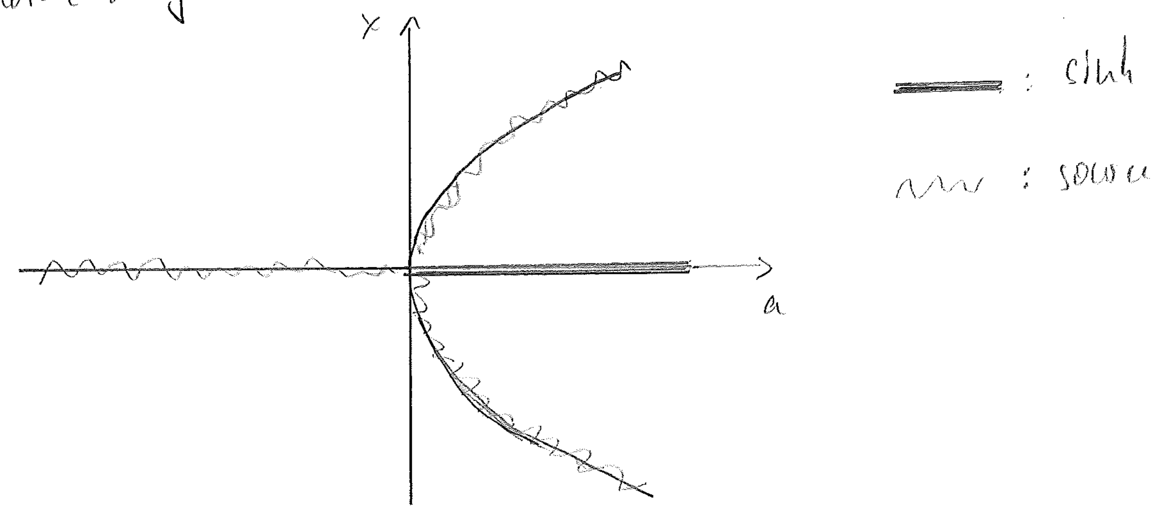
$a > 0$:



$x = 0$ source for $a < 0$, sink for $a > 0$

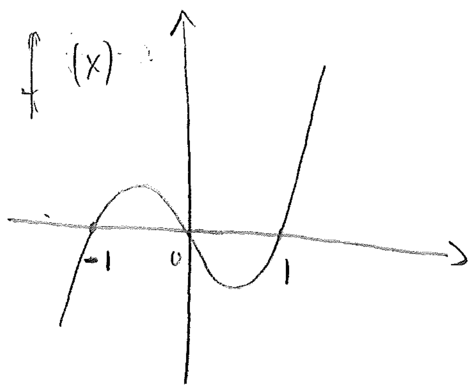
$x = \pm\sqrt{a}$ sources for $a > 0$

bifurcation diagram



this is a pitch fork bifurcation
(a central equilibrium changes stability
and two new equilibria of opposite
stability than the central one are
growing out of the central one)

(c) $x' = x^3 - x + a$
$=: f(x) + a$



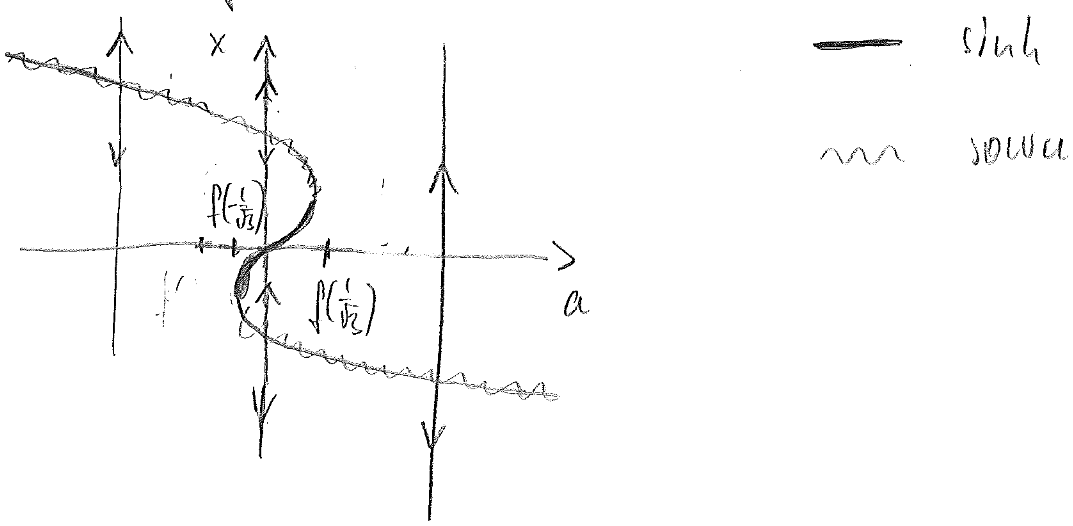
local extrema:

$f'(x) = 0$

$\Leftrightarrow \quad 3x^2 - 1 = 0$

$\Leftrightarrow \quad x = \pm\sqrt{\frac{1}{3}}$ (min/max)

bifurcation diagram



saddle-node bifurcations at $a = f(\frac{1}{\sqrt{3}})$ and $a = f(-\frac{1}{\sqrt{3}})$ (two equilibria of opposite stability collide and cease to exist, or two equilibria of opposite stability are born out of "nothing")

2.

$$X' = \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix} X$$

$$A := \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix}$$

$\det A = b^2$

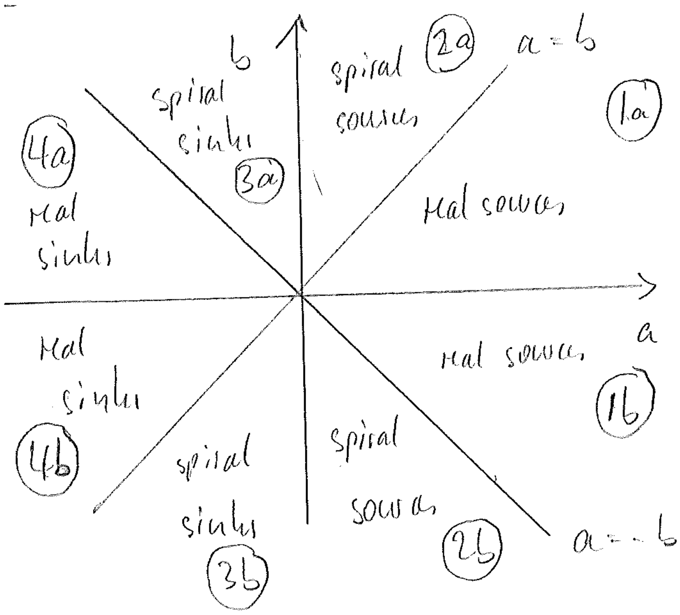
$\operatorname{tr} A = 2a$

eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}\right)$$

$$= \frac{1}{2}\left(2a \pm \sqrt{4a^2 - 4b^2}\right)$$

$$= a \pm \sqrt{a^2 - b^2}$$



canonical forms:

real sinks/sources

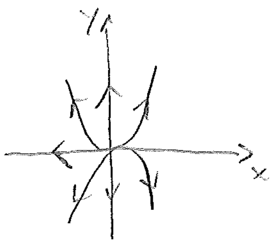
$$\vec{y}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{y} = \begin{pmatrix} a+\sqrt{a^2-b^2} & 0 \\ 0 & a-\sqrt{a^2-b^2} \end{pmatrix} \vec{y}$$

spiral sinks/sources

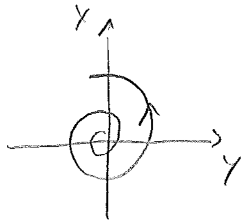
$$\vec{y}' = \begin{pmatrix} Re\,\lambda & Im\,\lambda \\ -Im\,\lambda & Re\,\lambda \end{pmatrix} \vec{y} = \begin{pmatrix} a & \sqrt{b^2-a^2} \\ -\sqrt{b^2-a^2} & a \end{pmatrix} \vec{y}$$

phase portraits

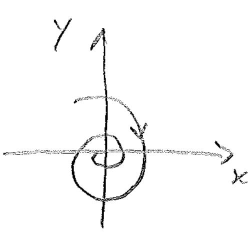
(1)



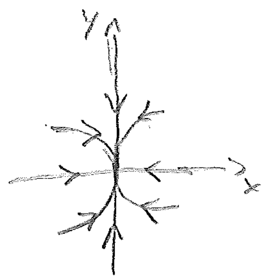
(2)



(3)



(4)



3. (a) $\frac{d}{dt} \tilde{V}(X(t)) = \nabla \tilde{V}(X(t)) \cdot X'(t) = -\|\nabla \tilde{V}(X(t))\|^2$

< 0 if $\nabla \tilde{V}(X(t)) \neq 0$,

i.e. if $X(t)$ is not an equilibrium

(b) An equilibrium point X_0 is asymptotically stable if for any neighb. $\mathcal{O}$ of X_0 there exists a neighb. $\hat{\mathcal{O}}$ such that f.a. $X \in \hat{\mathcal{O}}$ it holds that $\phi_t(X) \in \mathcal{O}$ f.a. $t \geq 0$.

(c) X^* isolated minimum of $\tilde{V} \Rightarrow \exists\, U$ open neighb. of X^* such that $V(X) > \tilde{V}(X^*)$ f.a. $X \in U \setminus \{X^*\}$

Set $\tilde{V}: U \to \mathbb{R}$, $\tilde{V}(X) := V(X) - \tilde{V}(X^*)$.

Then $\tilde{V}$ is strict Lyapunov function.

We can conclude that X^* is asymptotically stable and U is in the basin of attraction of X^*.

(d) The linearization at $\bar{X}^*$ is given by

$$\bar{Y}' = A\bar{Y} \quad \text{where} \quad A := -\text{Hess}\,\bar{V}(\bar{X}^*),$$

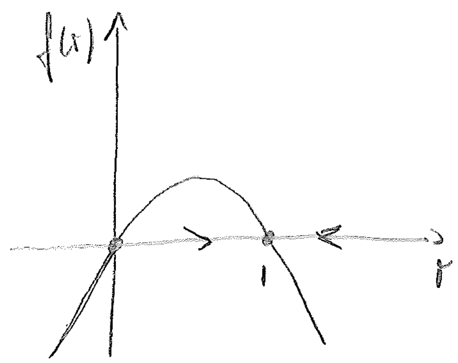
i.e. minus the Hessian matrix of $\bar{V}$ at $\bar{X}^*$.

~~As A is~~ ~~real~~ we know that it has real eigenvalues.

If A has no vanishing eigenvalues we know from the Hartman-Grobman Th^m (Linearization Th^m) that $\bar{X}' = -\nabla\bar{V}$ is near $\bar{X}^*$ conjugate to its linearization.

4. $r' = r - r^2 =: f(r)$

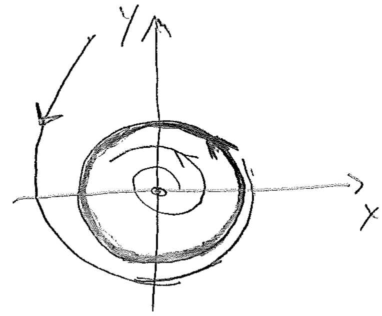
$\theta' = 1$



$x = r\cos\theta$

$y = r\sin\theta$

phase portrait:



- invariant circle $x^2+y^2=1$
- source at the origin

(a) For $(x,y) \in \mathbb{R}^2$ with

(i) $0 < x^2+y^2 < 1$, the α limit set is the origin and the ω limit set is the invariant circle

(ii) $1 < x^2+y^2$, the ω limit set is the invariant circle ("the α limit set is infinity")

(b) Poincaré - Bendixson Th^m: $P \subset \mathbb{R}^n$ compact and $X' = F(X)$ for $X \in P \subset \mathbb{R}^n$ positively invariant. Suppose that P contains only finitely many equilibria. Then the limit set of a point $X \in P$ is

(i) an equilibrium point or

(ii) a finite union of homoclinic or heteroclinic connections, or

(iii) a periodic orbit (limit cycle)

Application to the present case:

Choose $P := \{ (x,y) \in \mathbb{R}^2 \mid \frac{1}{4} \leq x^2 + y^2 \leq 4 \}$

which is the annulus with inner radius $\frac{1}{2}$ and outer radius 2.

$\Rightarrow$ P is positively invariant and P contains no equilibria.

By Poincaré-Bendixson Th^m we conclude that P must contain a limit cycle.

5. (a) let $f_{\lambda_0} =: f$.

$\Rightarrow |f'(x^*)| =: \nu < 1$. $\Rightarrow$ Choose, $1 > K > \nu$

$\Rightarrow \exists$ neighb. I of x^* s.t. $|f'(x)| < K$

f.a. $x \in I$. (as f' is cont.)

let $x \in I \setminus \{x^*\}$

$$\Rightarrow \frac{f(x) - x^*}{x - x^*} = \frac{f(x) - f(x^*)}{x - x^*} = f'(c_1) \text{ for some } c_1 \in I$$

(by Mean Value Th^{m}).

$$\Rightarrow |f(x) - x^*| = |f'(c_1)|\,|x - x^*|$$
$$< K\,|x - x^*|$$

similarly:

$$\frac{f^2(x) - x^*}{f(x) - x^*} = \frac{f(f(x)) - f(x^*)}{f(x) - x^*} = f'(c_2) \text{ for some } c_2 \in I$$

(note that $f(x) \in I$)

$$\Rightarrow |f^2(x) - x^*| = |f'(c_2)|\,|f(x) - x^*|$$
$$< K^2\,|x - x^*|$$

inductively

$$|f^n(x) - x^*| < K^n\,|x - x^*|$$
$$\rightarrow 0 \text{ as } n \to \infty.$$

$$\Rightarrow f^n(x) \to x^* \text{ as } n \to \infty \text{ f.a. } x \in I$$

(b) $f_\lambda : \mathbb{R} \to \mathbb{R}$ continuously differentiable w.r.t. x and λ.
$x \mapsto f_\lambda(x)$

fixed points are given implicitly by

$$g(\lambda, x) = f_\lambda(x) - x = 0$$

We have $\frac{\partial}{\partial x} g(\lambda_0, x^*) = f'_{\lambda_0}(x^*) - 1 \neq 0$ as $|f'_{\lambda_0}(x^*)| < 1$.

By the Implicit Function Theorem there exists
a $J \subset \mathbb{R}$ neighb. of λ_0, $I \subset \mathbb{R}$ neighb. of x^*
and $h : J_1 \to I_1$ such that $h(\lambda_0) = x^*$
and $f_\lambda(x) = x \iff x = h(\lambda)$,
f.a. $(x, \lambda) \in I_1 \times J_1$

As $|f''_{\lambda_0}(x^*)| > 1$ it follows that x^* is a source.

As f is continuously differentiable w.r.t. x and λ
it follows that $|f'_\lambda(x)| > 1$ for a
neighb. $I_2 \times J_2$ of (x^*, λ_0)

Choose $I = I_1 \cap I_2$ and $J = J_1 \cap J_2$